Constitutive Modeling of Multistage Creep Damage in Isotropic and Transversely Isotropic Alloys With Elastic Damage

In the pressure vessel and piping and power industries, creep deformation has continued to be an important design consideration. Directionally solidified components have become commonplace. Creep deformation and damage is a common source of component failure. A considerable effort has gone into the study and development of constitutive models to account for such behavior. Creep deformation can be separated into three distinct regimes: primary, secondary, and tertiary. Most creep damage constitutive models are designed to model only one or two of these regimes. In this paper, a multistage creep damage constitutive model is developed and designed to model all three regimes of creep for isotropic materials. A rupture and critical damage prediction method follows. This constitutive model is then extended for transversely isotropic materials. In all cases, the influence of creep damage on general elasticity (elastic damage) is included. Methods to determine material constants from experimental data are detailed. Finally, the isotropic material model is exercised on tough pitch copper tube and the anisotropic model on a Ni-based superalloy. [DOI: 10.1115/1.4005946]

Keywords: continuum damage mechanics (CDM), Kachanov, Rabotnov, Norton power law, McVetty time-hardening, coupled creep damage

1 Introduction

Creep deformation is a major failure mode in the pressure vessel and piping industry. Creep deformation is defined in three distinct stages: primary, secondary, and tertiary, as depicted in Fig. 1. During the primary creep regime, dislocations slip and climb. Eventually, a saturation of dislocation density coupled with recovery mechanics in balance form the secondary creep regime. Finally, the tertiary creep regime is observed where grain boundaries slide, voids form, and coalescence, leading to rupture.

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Depending on the material composition, component, and service condition, each regime can become a critical design requirement. The earliest efforts to model creep focused on the short term creep strain observed during the primary creep regime [1]. Later efforts focused on the balanced behavior observed in the secondary creep regime [2], and more modern efforts focus on the end of life behavior observed during the tertiary creep regime [3,4].

While many authors focus on individual creep regimes, only a few authors have produced fully developed multistage models, i.e., a model that predicts the deformation for all three creep regimes [5]. Little work has been done for modeling anisotropic materials [6,7]. To that end, a multistage creep damage constitutive model is developed [8]. It is initially designed for isotropic materials and then extended for transversely isotropic materials. Rupture and critical damage prediction methods are included. Elastic damage is implemented using relevant theories. Analytical methods to determine the material constants associated with each regime of creep are provided. Creep deformation data obtained

from literature are used to verify the applicability of the isotropic and transversely isotropic formulations.

2 Consitutive Model

Two forms of the multistage creep damage constitutive model with elastic damage are proposed. Initially, an isotropic form is derived. Then, using the creep potential hypothesis, a tensorial transversely isotropic model is developed [9].

2.1 Isotropic Material. The isotropic multistage creep damage model comprised two strain rate equations separated into primary, \dot{e}_{pr} , and secondary, \dot{e}_{sc} , portions

$$\dot{\varepsilon}_{cr} = \dot{\varepsilon}_{pr} + \dot{\varepsilon}_{sc} \tag{1}$$

The primary creep strain equation is a power law extension of the McVetty time-hardening primary creep law [10] as follows

$$\varepsilon_{pr} = A_{pr}\bar{\sigma}^{n_{pr}}(1 - e^{-qt}) \tag{2}$$

where A_{pr} , n_{pr} , and q are primary creep material properties, which vary with temperature, and $\bar{\sigma}$ is the von Mises equivalent stress. Further examination shows that the two terms in the equation are the constant stress deformations of Voigt and Maxell elements, respectively [11]. Differentiation furnishes the primary creep strain rate as

$$\dot{\epsilon}_{pr} = qA_{pr}\bar{\sigma}^{n_{pr}}e^{-qt} \tag{3}$$

Variations of this equation exist for strain-hardening and combined time-strain-hardening [11].

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Fig. 1 Creep deformation

To predict secondary and tertiary creep, the Kachanov– Rabotnov coupled creep-damage model is employed [3,4]. The underlying foundation of this model is the concept of effective stress and damage

$$\tilde{\sigma} = \sigma \frac{A_0}{A_{\text{net}}} = \frac{\sigma}{\left(1 - \frac{A_0 - A_{\text{net}}}{A_0}\right)} = \frac{\sigma}{(1 - \omega)}$$
(4)

where physical material damage (approximated as net area reduction) is equivalent to an effective increase in the stress in the undamaged continuum. The secondary creep strain rate and damage evolution equations of the Kachanov–Rabotnov [3,4] model are

$$\dot{\varepsilon}_{sc} = A \left(\frac{\bar{\sigma}}{1 - \omega} \right)^n \tag{5}$$

$$\dot{\omega} = \frac{M\bar{\sigma}^{\chi}}{(1-\omega)^{\phi}}, \quad 0 \le \omega < 1 \tag{6}$$

where the creep strain rate is equivalent to Norton's power law for secondary creep [2] with the same *A* and *n* secondary creep constants, $\bar{\sigma}$ is von Mises stress, and *M*, χ , and ϕ are tertiary creep damage constants. Tertiary creep arises within the secondary creep equation due to the coupling damage term.

Using the principle of strain equivalence, elastic damage can be introduced in linear elasticity in the following nonrigorous form

1D

$$\tilde{E} = E_0(1 - \omega)$$
 where $0 \le \omega \le 1$
2D
 $\tilde{E} = E_0(1 - \omega)$ (7)
 $\nu = \nu_0 \sqrt{(1 - \omega)} / \sqrt{(1 - \omega)}$
 $G = G_0(E/E_0)(\nu/\nu_0)$

where E_0 is Young's modulus, ν_0 is Poisson's ratio, and G_0 is the shear modulus [12]. Sidoroff has shown that an alternative more robust form can be derived using the hypothesis of elastic energy equivalence [13]. A number of authors have extended this for three-dimensional anisotropic damage accumulation [14].

Finally, the total strain can be added together as follows (for a 1D model)

$$\varepsilon = = \frac{\sigma}{\tilde{E}} + \dot{\varepsilon}_{pr} \Delta t + \dot{\varepsilon}_{sc} \Delta t \tag{8}$$

where Δt represents the time increment.

A rupture prediction can be found by integration of the damage evolution [Eq. (6)] as follows

$$(1-\omega)^{\phi}d\omega = M\sigma_{r}^{\chi}dt$$
$$-\frac{(1-\omega)^{\phi}}{1+\phi}\Big|_{\omega_{o}}^{\omega} = M\sigma_{r}^{\chi}\Big|_{t_{o}}^{t}$$
(9)

where stress and temperature are constant. Assuming initial time, t_o , and initial damage, ω_o , equal zero leads to

$$t = \left[1 - (1 - \omega)^{\phi + 1}\right] \left[(\phi + 1)M\sigma_r^{\chi}\right]^{-1}$$
(10)

$$\omega(t) = 1 - \left[1 - (\phi + 1)M\sigma_r^{\chi} t\right]^{\frac{1}{\phi+1}}$$
(11)

To predict rupture time, t_r , the critical damage, ω_{cr} , must be given. Critical damage is assumed to be some value less than unity.

2.2 Transversely Isotropic Material. To account for multiaxial states of stress and orthotropic material behavior, a tensorial formulation is desired.

Using the creep potential hypothesis, a general flow rule can be developed using a potential function. For example, using the von Mises yield criterion, the following flow rule is obtained

$$d\varepsilon_{ij,cr} = \frac{3}{2} \frac{d\varepsilon^*}{\sigma^*} \frac{d\psi}{d\sigma_{ij}}$$
(12)

where $d\varepsilon^*$ is the equivalent strain increment, σ^* is the equivalent stress, and $\psi(\sigma_{ij}) = \sigma^{*2}/3$ is the von Mises plastic potential function [9].

A potential function for anisotropic materials is needed. The Hill's potential theory is an extension of the von Mises yield criterion that takes into account anisotropic yield of materials and takes the following form

$$\sigma_{\text{Hill}} = \sqrt{\mathbf{s}^T \mathbf{M} \mathbf{s}}$$

$$\mathbf{s} = VEC(\boldsymbol{\sigma})$$

$$\mathbf{M} = \begin{bmatrix} G + H & -H & -G & 0 & 0 & 0 \\ -H & F + H & -F & 0 & 0 & 0 \\ -G & -F & F + G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2N & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix}$$
(13)

where σ_{Hill} is Hill's equivalent stress, **s** is the 6 × 1 vector form of the Cauchy stress tensor, σ and **M** is the Hill compliance tensor [15] consisting of the *F*, *G*, *H*, *L*, *M*, and *N* unitless material constants that can be obtained from creep tests [16]. It should be noted that Hill's equivalent stress reverts to von Mises when

$$F = G = H = \frac{1}{2}$$

$$L = M = N = \frac{3}{2}$$
(14)

Using Hill's potential function and the creep potential hypothesis, a general flow rule of the time-hardening primary creep law [Eq. (3)] is formulated for transversely isotropic material in the following form

$$\dot{\mathbf{e}}^{pr} = q_{\text{aniso}} A_{\text{anisoP}} \sigma_{\text{HillP}}{}^{n_{\text{anisoP}}} e^{-q_{\text{aniso}}t} \frac{\mathbf{M}_{P} \mathbf{s}}{\sigma_{\text{HillP}}}$$
(15)

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where A_{anisoP} , n_{anisoP} , and q_{aniso} are material properties, which vary with temperature and \mathbf{M}_P is a unique Hill compliance tensors such that the unique Hill's equivalent stress σ_{HillP} exists.

This approach is repeated for the Kachanov–Rabotnov secondary creep strain rate [Eq. (5)], yielding the following

$$\dot{\mathbf{e}}^{cr} = A_{\text{aniso}} \tilde{\sigma}_{\text{Hill}}{}^{n_{\text{aniso}}} \frac{\mathbf{Ms}}{\sigma_{\text{Hill}}}$$
(16)

where A_{aniso} and n_{aniso} are anisotropic material properties, which vary with temperature, **M** is the Hill compliance tensor with six constants, and $\tilde{\sigma}_{\text{Hill}}$ is the effective Hill's equivalent stress, a function of the effective stress vector, $\tilde{\mathbf{s}}$, which will be further examined later [17].

A problem becomes apparent when attempting to implement the general flow rule in the damage evolution [Eq. (6)]. The use of the variable current damage, ω , in the denominator prevents the simple approach used previously. Damage evolution must be split into two submatrices as follows

$$\mathbf{b} = M_{\text{aniso}} \sigma_{\text{Hillb}}^{\chi_{\text{aniso}}} \frac{\mathbf{M}_b \mathbf{s}}{\sigma_{\text{Hillb}}} \quad \lambda = \phi_{\text{aniso}} \frac{\mathbf{M}_\lambda \mathbf{s}}{\sigma_{\text{Hill}\lambda}} \tag{17}$$

and combined in the following damage rate vector

$$\dot{\omega}_i = \frac{|\mathbf{b}_i|}{(1 - \omega_i)^{|\lambda_i|}} \tag{18}$$

where M_{aniso} , χ_{aniso} , and ϕ_{aniso} are anisotropic tertiary creep damage constants. The tensors \mathbf{M}_b and \mathbf{M}_λ are unique Hill compliance tensors of the same form as Eq. (13) such that unique Hill equiva-

lent stresses $\sigma_{\text{Hill}\lambda}$ and $\sigma_{\text{Hill}\lambda}$ may arise. The six constants required for each tensor can be found from creep tests [16].

Rabotnov [18] proposed a generalized fourth order tensor that relates the effective and Cauchy stress vectors

$$\tilde{\mathbf{s}} = \Omega(\boldsymbol{\omega}) \cdot \mathbf{s} \tag{19}$$

where Ω , the damage applied tensor, is a function of damage. Extending the fundamental effective stress approach, [Eq. (4)], the effective stress vector becomes

$$\tilde{\mathbf{s}} = (\mathbf{I} - \mathbf{D})^{-1} \mathbf{s}, \quad \mathbf{D} = \operatorname{diag}(\omega_1, \omega_2, ..., \omega_n)$$

$$\mathbf{D} = \begin{bmatrix} \omega_1 & 0 & 0 & 0 \\ 0 & \omega_2 & 0 & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \omega_n \end{bmatrix}$$
(20)

where ω is the damage vector and **I** and **D** represent a fourth rank identity tensor and damage tensor, respectively.

General linear elasticity can be described by the Hooke's law as

$$s = Ce$$

$$e = Ss$$
(21)

where s and e are the Cauchy stress and strain tensors and C and S are the stiffness and compliance tensors, respectively. Taking cues from the principle of strain equivalence and the hypothesis of elastic energy equivalence, the isotropic elastic damage, Eq. (7), is extended for transversely isotropic materials into the form of

$$\mathbf{e}_{1} = \tilde{\mathbf{S}}_{\mathbf{S}}$$

$$\begin{bmatrix} e_{11} \\ e_{22} \\ e_{33} \\ e_{12} \\ e_{23} \\ e_{31} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_{p}(1-\omega_{1})} & -\frac{\nu_{p}}{E_{p}\sqrt{1-\omega_{1}}\sqrt{1-\omega_{2}}} & -\frac{\nu_{zp}}{E_{z}\sqrt{1-\omega_{3}}\sqrt{1-\omega_{1}}} & 0 & 0 & 0 \\ -\frac{\nu_{p}}{E_{p}\sqrt{1-\omega_{1}}\sqrt{1-\omega_{2}}} & \frac{1}{E_{p}(1-\omega_{2})} & -\frac{\nu_{zp}}{E_{z}\sqrt{1-\omega_{3}}\sqrt{1-\omega_{1}}} & 0 & 0 & 0 \\ -\frac{\nu_{pz}}{E_{p}\sqrt{1-\omega_{1}}\sqrt{1-\omega_{3}}} & -\frac{\nu_{pz}}{E_{p}\sqrt{1-\omega_{1}}\sqrt{1-\omega_{3}}} & \frac{1}{E_{z}(1-\omega_{3})} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1+\nu_{p}}{E_{p}(1-\omega_{4})} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2G_{zp}(1-\omega_{5})} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2G_{zp}(1-\omega_{5})} \end{bmatrix} \begin{bmatrix} s_{11} \\ s_{22} \\ s_{33} \\ s_{12} \\ s_{31} \end{bmatrix}$$

where the Young's moduli, Poisson's ratios, and shear modulus are E_p , E_z , ν_p , ν_{zp} , ν_{pz} , and G_{zp} , respectively.

Finally, the total strain in vector form can be added together as follows

$$\varepsilon_i = \tilde{S}_{ij} s_j + \dot{\varepsilon}_i^{pr} \Delta t + \dot{\varepsilon}_i^{sc} \Delta t \tag{23}$$

where Δt represents the time increment.

Similar to the isotropic approach, a rupture prediction can be found by integration of the damage evolution [Eq. (18)] leading to

$$t_{r_{i}} = \left[1 - (1 - \omega_{cr})^{|\lambda_{i}|+1}\right] [(|\lambda_{i}| + 1) \cdot |b_{i}|]^{-1}$$

$$t_{r} = \min\{t_{r_{i}}\}$$

$$\omega_{cr_{i}}(t_{r_{i}}) = 1 - [1 - (|\lambda_{i}| + 1) \cdot |b_{i}| \cdot t_{r_{i}}]^{\frac{1}{|\lambda_{i}|+1}}$$

$$\omega_{r} = \min\{\omega_{cr_{i}}\}$$
(24)

where the stress tensor and temperature are constant. The rupture time and critical damage, t_r and ω_r , are the minimum values found in the respective vectors. Again, a value of critical damage

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is required to produce rupture predictions. It should be noted that this rupture prediction method is based on limited data. A more robust methodology would be based on a creep rupture loci that is formed from uniaxial and multiaxial creep test data at various orientations.

3 Determination of Creep Material Properties

There are ten material constants required to model elastic, primary, secondary, and tertiary creep for an isotropic material. The number is increased to 30 when dealing with transversely isotropic materials. In this section, the method to determine the isotropic creep constants is formulated. Then, an analytical method for transversely isotropic material properties is outlined.

3.1 Isotropic Material. The isotropic constitutive model requires ten (when $n_{pr} = n$ is assumed, this reduces to nine) material constants (Table 1). The creep material constants can be determined from a single constant load and temperature creep experiment. The constants are found in the following order: secondary creep, primary creep, and, finally, tertiary creep damage constants.

3.1.1 Secondary Creep. Secondary creep is characterized by a balance between strain-hardening and recovery mechanics, which leads to a steady strain rate. This steady rate $\dot{\epsilon}_{min}$ is described as the minimum creep strain. The derivative (via finite difference) of strain versus time can furnish a value of the minimum strain rate.

Assuming that damage is zero and replacing \dot{e}_{sc} with the minimum creep strain rate \dot{e}_{min} , the Kachanov–Rabotnov strain rate [Eq. (5)] equates to

$$\dot{\varepsilon}_{\min} = A\bar{\sigma}^n \tag{25}$$

where A and n are the secondary creep coefficient and exponent, respectively, and $\bar{\sigma}$ is the equivalent stress. The secondary creep coefficient and exponent can be determined from uniaxial creep tests by rearranging Eq. (25) into the following form

$$\ln \dot{\varepsilon}_{\min} = \ln A + n \ln \bar{\sigma} \tag{26}$$

Plotting the natural log of the minimum creep strain versus the log of equivalent stress, the requisite linear function can be found. A variation of this method can be used to eliminate temperature-dependence from the constants [19].

3.1.2 Primary Creep. The primary creep constants can be determined using the reverse creep approach [10,11]. Assume the following creep strain

$$\varepsilon_{cr} = \varepsilon_{pr} + \varepsilon_{sc}, \quad t < t_0 \tag{27}$$

where t_0 is the time at which the minimum creep strain rate, \dot{e}_{min} is reached. Rearranging Eq. (27) and applying the primary creep strain law [Eq. (2)] produce

$$\varepsilon_{pr} = \varepsilon_{cr} - \varepsilon_{sc} = A_{pr}\bar{\sigma}^{n_{pr}}(1 - e^{-qt}), \quad t < t_0$$

where $\varepsilon_{pr}^{\max} = \varepsilon_{cr} - \dot{\varepsilon}_{\min}\Delta t = A_{pr}\bar{\sigma}^{n_{pr}}, \quad t = t_0$ (28)

where $\Delta t = t_0 - 0$ is the time increment. The A_P and n_P primary creep coefficient and exponent can be found from uniaxial creep tests using the maximum primary creep strain, ε_{pr}^{\max} in the following form

$$\ln \varepsilon_{pr}^{\max} = \ln(\varepsilon_{cr} - \dot{\varepsilon}_{\min}\Delta t) = \ln A_{pr} + n_{pr}\ln\sigma$$
(29)

Plotting the natural log of the maximum primary creep strain versus the natural log of equivalent stress, the requisite linear function can be found. Similarly, assume the following creep strain rate

$$\dot{\varepsilon}_{cr} = \dot{\varepsilon}_{pr} + \dot{\varepsilon}_{sc}, \quad t < t_0 \tag{30}$$

Rearranging Eq. (30), applying the primary creep strain rate [Eq. (3)], and replacing \dot{e}_{sc} with the minimum creep strain rate \dot{e}_{min} produce

$$\dot{\varepsilon}_{pr} = \dot{\varepsilon}_{cr} - \dot{\varepsilon}_{\min} = qA_{pr}\bar{\sigma}^{n_{pr}}e^{-qt}, \quad t < t_0$$
(31)

where there are two methods to find the q constant. The above equation can be directly implemented in a nonlinear equation solver or implemented in regression analysis and solved for the q constant.

3.1.3 Tertiary Creep Damage. The authors have proposed an analytical method to determine the tertiary creep damage constants for the Kachanov–Rabotnov constitutive model [20]. First, the creep strain rate is found from experimental data using finite differencing

$$\dot{\varepsilon}_{cr}^{i+1} = \frac{\varepsilon_{cr}^{i+1} - \varepsilon_{cr}^{i}}{t_{i+1} - t_{i}}$$
(32)

Algebraic manipulation of the Kachanov–Rabotnov creep strain rate [Eq. (5)] leads to the following

$$\omega(\dot{\varepsilon}_{cr}) = \frac{\left(\frac{\dot{\varepsilon}_{cr}}{A}\right)^{1/n} - \sigma}{\left(\frac{\dot{\varepsilon}_{cr}}{A}\right)^{1/n}}$$
(33)

At time zero, the values of damage found will be high. This is attributed to the high creep strain rate observed in the primary creep regime. The Kachanov–Rabotnov constitutive model does not account for the strain hardening of primary creep. The damage data should be modified such that damage is set to zero until the minimum creep strain rate is reached. Next, the rupture prediction model [Eq. (10)] with time set to t_r is algebraically manipulated to find an *M* constraint as follows

$$M = \frac{1 - (1 - \omega_{cr})^{\phi + 1}}{(\phi + 1)\bar{\sigma}^{\chi} t_r}$$
(34)

where the critical damage, ω_{cr} , is equal to the final value found from finite differencing the experimental data, and t_r is the rupture time. This *M* constraint is introduced into the damage prediction equation [Eq. (11)], furnishing

$$\omega(t) = 1 - \left[\frac{t}{t_r} \left[(1 - \omega_r)^{\phi + 1} - 1 \right] + 1 \right]^{\frac{1}{\phi + 1}}$$
(35)

where equivalent stress, $\bar{\sigma}$, and both the *M* and χ tertiary creep damage constants are eliminated. The constants *M* and χ are dependent while ϕ is independent. When using this approach, the constant χ should be chosen arbitrarily. The constant *M* should be found using the constraint equation [Eq. (34)]. These steps produce a well-defined equation designed to satisfy experimental conditions. Finally, using suitable regression analysis software, the modified damage evolution equation [Eq. (35)] can be written as a user-defined equation and the tertiary creep damage constants are determined.

3.2 Transversely Isotropic Constants. The transversely isotropic constitutive model requires 30 (when $n_{LP} = n_L$, $n_{TP} = n_T$, and $n_{45P} = n_{45}$ are assumed this reduces to

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27) material constants (Table 1). The creep material constants can be determined from three constant load and temperature creep experiments: longitudinal-, transverse-, and 45 deg-grain orientations. The constants are found in the following order: secondary creep, primary creep, and, finally, tertiary creep damage constants.

3.2.1 Secondary Creep. The anisotropic secondary creep constants, A_{aniso} and n_{aniso} , and the Hill constants F, G, H, L, M, and Nfound in Eq. (16) can be derived from three constant load and temperature specimen [17]. The isotropic secondary creep constants Aand n are determined for each experiment [21] and are of the form

$$\dot{\vec{k}}_{33}^{\min} = A_L \sigma^{n_L} \\
\dot{\vec{k}}_{33}^{\min} = A_T \sigma^{n_T} \\
\dot{\vec{k}}_{33}^{\min} = A_{45 \ \deg} \sigma^{n_{45 \ \deg}}$$
(36)

where $\dot{\epsilon}_{33}^{\min}$ describes the minimum creep strain rate found on the load axis x_3 of each specimen.

To determine the anisotropic secondary creep constants, damage evolution [Eq. (18)] is disabled by setting ω and $\dot{\omega}$ equal. The creep strain rate equation [Eq. (16)] can then be rewritten as follows

$$\mathbf{e}^{cr} = A_{\text{aniso}} \sqrt{\mathbf{s}^{T} \mathbf{T} \mathbf{M} \mathbf{T}^{T} \mathbf{s}}^{n_{\text{aniso}}} \frac{\mathbf{T} \mathbf{M} \mathbf{T}^{T} \mathbf{s}}{\sqrt{\mathbf{s}^{T} \mathbf{T} \mathbf{M} \mathbf{T}^{T} \mathbf{s}}}$$
$$\mathbf{M} = \begin{bmatrix} G + H & -H & -G & 0 & 0 & 0 \\ -H & F + H & -F & 0 & 0 & 0 \\ -G & -F & F + G & 0 & 0 & 0 \\ 0 & 0 & 0 & 2N & 0 & 0 \\ 0 & 0 & 0 & 0 & 2L & 0 \\ 0 & 0 & 0 & 0 & 0 & 2M \end{bmatrix}$$
(37)

where **T** represents a material orientation transformation tensor about the x_I axis. The desired constants can be found by rotating the material orientation and/or state of stress.

For an L-oriented specimen, the longitudinal behavior is aligned with the x_3 normal while the transverse behavior is observed on x_1 and x_2 normals. In the **T** tensor, $\alpha = 0$ deg, leading to a creep strain rate in the x_3 normal of

$$\dot{\varepsilon}_{33} = A_L \sigma^{n_L} = A_{aniso} (G+F)^{\frac{n_{aniso}+1}{2}} \sigma^{n_{aniso}}_{33}$$
 (38)

where it is assumed that F = G and G is chosen arbitrarily. Setting G to 0.5 results in A_{aniso} and n_{aniso} equaling A_L and n_L , respectively.

For an T-oriented specimen, $\alpha = 90$ deg, thus the creep strain rate in the x_3 normal resolves into the following

$$\dot{\varepsilon}_{33} = A_T \sigma^{\mathbf{n}_T} = A_{\text{aniso}} (G+H)^{\frac{n_{\text{aniso}}+1}{2}} \sigma^{n_{\text{aniso}}}_{33}$$
 (39)

where rearranging Eq. (38) to solve for A_{aniso} and applying it into Eq. (39) allow *H* to be found

$$H = (2t_1 - 1)G$$

$$t_1 = \left(\frac{A_T \sigma^{\mathbf{n}_T}}{A_L \sigma^{\mathbf{n}_L}}\right)^{2/(n_{\text{aniso}} + 1)}$$
(40)

where t_1 is a unitless anisotropy factor.

For a 45 deg-oriented specimen, $\alpha = 45$ deg, thus, the creep strain rate in the x_3 normal resolves into the following

$$\dot{\varepsilon}_{33} = A_{45 \text{ deg}} \sigma^{n_{45 \text{ deg}}} = A_{\text{aniso}} (0.25G + 0.25H + 0.5M)^{\frac{\text{canso} - 2}{2}} \sigma^{n_{\text{aniso}}}_{33}$$
(41)

Using Eq. (39) to solve for A_{aniso} and applying it into Eq. (41) allow *M* to be found

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$$M = G(4t_2 - t_1)$$

$$t_2 = \left(\frac{A_{45 \text{ deg}}\sigma^{n_{45 \text{ deg}}}}{A_L\sigma^{n_L}}\right)^{2/(n_{\text{aniso}} + 1)}$$
(42)

where, due to symmetry, it is assumed that L = M, and t_2 is a unitless anisotropy factor. To determine the final constant, N, a symbolic plane stress rotation is applied. This approach uses the concept of equivalent stress. Initially, the state of stress is set as

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_0 & 0 & 0\\ 0 & -\sigma_0 & 0\\ 0 & 0 & \sigma_{33} \end{bmatrix}$$
(43)

where the Hill equivalent stress resolves to the following

$$\pi_{\text{Hill}} = \sqrt{2G(\sigma_{33} + \sigma_0)^2 + 4H\sigma_0^2}$$
(44)

In the case where the state of stress is rotated by 45 deg about the x_3 axis, pure shear stress develops in the x_1 - x_2 plane of the form

$$\sigma' = \mathbf{Q}\sigma\mathbf{Q}^T = \begin{bmatrix} 0 & -\sigma_0 & 0\\ -\sigma_0 & 0 & 0\\ 0 & 0 & \sigma_{33} \end{bmatrix}$$
(45)

where the Hill equivalent stress resolves to the following

$$\sigma_{\rm Hill} = \sqrt{2G\sigma_{33}^2 + 2N\sigma_0^2} \tag{46}$$

By equating Eqs. (44)–(46), the N constant can be determined.

$$N = (4t_1 - 1)G \tag{47}$$

3.2.2 Primary Creep. The anisotropic primary creep constants A_{anisop} , n_{anisop} , and q_{aniso} , and the Hill constants for the \mathbf{M}_P compliance can similarly be derived from three constant load and temperature specimen [17]. First, isotropic primary creep constants A_P , n_P , and q are determined from each experiment [21] and are of the form

$$\dot{\varepsilon}_{pr,33} = q_L A_{LP} \bar{\sigma}^{n_{LP}} e^{-q_L t}
\dot{\varepsilon}_{pr,33} = q_T A_{TP} \bar{\sigma}^{n_{TP}} e^{-q_T t}
\dot{\varepsilon}_{pr,33} = q_{45} A_{45P} \bar{\sigma}^{n_{45P}} e^{-q_{45} t}$$
(48)

where $\dot{e}_{pr,33}$ is the primary creep strain rate found on the load axis x₃ of each specimen.

The same approach used to determined the secondary creep behavior furnishes the following

$$A_{\text{anisoP}} = A_{LP}, \quad n_{\text{anisoP}} = n_{LP}, \quad q_{\text{aniso}} = q_{L}$$

$$t_{1} = \left(\frac{q_{T}A_{TP}\bar{\sigma}^{n_{TP}}e^{-q_{T}t}}{q_{L}A_{LP}\bar{\sigma}^{n_{LP}}e^{-q_{L}t}}\right)^{2/(n_{\text{anisoP}}+1)}$$

$$t_{2} = \left(\frac{q_{45}A_{45P}\bar{\sigma}^{n_{45P}}e^{-q_{45}t}}{q_{L}A_{LP}\bar{\sigma}^{n_{LP}}e^{-q_{L}t}}\right)^{2/(n_{\text{anisoP}}+1)}$$
(49)

where the same relations for F, G, H, L, M, and N are used to find the constants for \mathbf{M}_P through the replacement of t_1 and t_2 .

3.2.3 Tertiary Creep Damage. The anisotropic tertiary creep damage constants, M_{aniso} , χ_{aniso} , and ϕ_{aniso} , and the Hill constants for the \mathbf{M}_b and \mathbf{M}_λ compliance can be similarly be derived from three constant load and temperature specimen [17]. The isotropic creep damage constants M_L , M_T , M_{45} , χ_{L} , χ_{T} , χ_{45} , ϕ_L , ϕ_T , and ϕ_{45} are determined using the isotropic approach.

As previously stated, the classic Kachanov-Rabotnov damage behavior was separated into two vectorized damage constants

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Table 1 List of multistage model constants

Material model	Isotropic constants	Transversely isotropic constants				
Elasticity	E, ν	$E_p, E_z, \nu_p, \nu_{zp}, \nu_{pz}, G_{zp}$				
Primary creep	A_{pr}, n_{pr}, q	$A_{LP}, A_{TP}, A_{45P}, n_{LP}, n_{TP}, n_{45P}, q_L, q_T, q_{45}$				
Secondary creep	A, n	$A_L, A_T, A_{45}, n_L, n_T, n_{45}$				
Tertiary creep	M, χ, ϕ	$M_L, M_T, M_{45}, \chi_L, \chi_T, \chi_{45}, \phi_L, \phi_T, \phi_{45}$				
damage						
Total	10	30				

Table 2 Creep deformation data for copper [23]

Temp., T (°C)	Stress, σ (MPa)	Min strain rate (%/h)	Rupture strain (%)	Rupture time (h)	Approx. critical damage
250	40	$\begin{array}{c} 2.3790 \times 10^{-5} \\ 9.0913 \times 10^{-5} \\ 1.8710 \times 10^{-4} \end{array}$	7.826	1112	0.3794
250	50		7.069	442.7	0.4058
250	60		7.184	233.6	0.3361

tensors [Eq. (17)]. To determine constants for the \mathbf{M}_b compliance tensor the numerator of Kachanov–Rabotnov damage evolution [Eq. (6)] was equated to the associated component of the **b** tensor. In the case of the \mathbf{M}_{λ} compliance tensor, the ϕ_i constants were directly related to the associated component of the λ tensor. With these changes, the same approach used to determined the secondary creep behavior furnished the following

$$M_{\text{aniso}} = M_L, \quad \chi_{\text{aniso}} = \chi_L, \quad \phi_{\text{aniso}} = \phi_L$$

$$\mathbf{M}_b \qquad \mathbf{M}_{\lambda}$$

$$t_1 = \left(\frac{M_T \sigma^{\chi_T}}{M_L \sigma^{\chi_L}}\right)^{2/(\chi_{\text{aniso}}+1)} \qquad t_1 = \left(\frac{\phi_T}{\phi_L}\right)^2 \qquad (50)$$

$$t_2 = \left(\frac{M_{45 \text{ deg}} \sigma^{\chi_{45 \text{ deg}}}}{M_L \sigma^{\chi_L}}\right)^{2/(n_{\text{aniso}}+1)} \qquad t_2 = \left(\frac{\phi_{45 \text{ deg}}}{\phi_L}\right)^2$$

where the same relations for *F*, *G*, *H*, *L*, *M*, and *N*are used to find the constants for \mathbf{M}_b and \mathbf{M}_{λ} through the replacement of t_1 and t_2 .

4 Results and Discussion

Both the isotropic and anisotropic constitutive models with elastic damage were implemented into a general-purpose finite element analysis (FEA) software, ANSYS. A USERMAT3D userprogrammable feature (UPF) is coded in FORTRAN. In USER-MAT3D, the strain increment, strain, and stress vectors are provided. An updated stress vector must be output. An input deck using the ANSYS parametric design language (APDL) has been created. In the input deck, a single element is used to approximate a uniaxial creep test. Appropriate displacement constraints are applied. Constants load and temperature boundary conditions are set. The input deck is flexible such that boundary conditions can be parametrically exercised. It should be noted that this constitutive model can be utilized in any appropriate FEM software package.

4.1 Copper—Isotropic Model. Copper was selected as the material to verify the isotropic constitutive model. Copper exhibits all three regimes of creep. Numerous studies on the creep behavior of copper have been performed [22,23].

Literature has provided creep deformation curves for tough pitch copper tube [23]. Creep deformation data are given in Table 2. The Young's modulus and Poisson ratio of tough pitch copper at 250 °C are 103 GPa and 0.31, respectively. Using the approach outlined in Sec. 3, the creep properties of copper were determined and are given in Table 3. Observation shows that $n = \chi$ for each stress level; therefore, the number of independent constants required (including elasticity) is reduced to nine. Creep deformation and damage evolution simulations of these experiments were performed and are shown in Figs. 2 and 3.

Experimental data are represented by symbols and simulations by lines. It is observed that the constitutive model accurately models the creep strain during each creep regime. The damage experimental points were obtained through analytical method [Eq. (33)]. This analytical method creates erroneous values during the

Table 3 Creep properties of copper

Stress, σ (MPa)	A_{pr} (MPa ^{-np})	n _{pr}	$q (h^{-1})$	$(\mathrm{MPa}^{-n}\mathrm{h}^{-1})$	п	$(\mathrm{MPa}^{-\chi}\mathrm{h}^{-1})$	χ	ϕ
40	1.3245×10^{-5}	1.6759	1/3	$2.6929 imes 10^{-11}$	3.8459	5.6077	3.8459	10.00
50	_	_	_	_	_	5.9831	_	_
60	-	_	_	_	_	5.5799	-	-



Fig. 2 Creep deformation of tough pitch copper tube: (*a*) primary and secondary regimes and (*b*) all regimes

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Fig. 3 Damage evolution of tough pitch copper tube: (a) damage and (b) Young's moduli

 Table 4
 Creep deformation data for Ni-based DS superalloy
 [24]

Orient, α (deg)	Temp., T (°C)	Stress, σ (MPa)	Min strain rate (h ⁻¹)	Rupture strain (%)	Rupture time (h)	Critical damage
0	870	250	$\begin{array}{c} 2.4889 \times 10^{-5} \\ 1.4869 \times 10^{-5} \\ 1.5378 \times 10^{-5} \end{array}$	4.464	901.4	0.375
45	870	250		2.530	955.9	0.500
90	870	250		3.376	890.9	0.478

primary creep regime that can be discounted. The damage evolution simulations closely match the experimental data once pass the primary creep regime. Figure 3(b) demonstrates elastic damage represented by reduction in the Young's moduli.

4.2 DS Ni-Based Superalloy—Anisotropic Model. A DS Ni-based superalloy was selected as the material to verify the anisotropic constitutive model. Literature provides very few studies on the creep behavior of DS Ni-based superalloys [19,24]. Creep deformation data for a DS Ni-based superalloy are given in Table 4.

Table 5 Creep properties of Ni-based DS superalloy

Orient, α (deg)	(MPa^{-np})	n _p	(h^{-1})	$(\mathrm{MPa}^{-n}\mathrm{h}^{-1})$	n	$(\mathrm{MPa}^{-\chi}\mathrm{h}^{-1})$	χ	Φ
0	2.8800×10^{-10}	3	1/8	1.5929×10^{-12}	3	1.5324×10^{-11}	3	2.889
45	1.4400×10^{-10}	3	1/7	9.5160×10^{-13}	3	$9.7973 imes 10^{-12}$	3	5.771
90	1.6000×10^{-10}	3	1/6	9.8417×10^{-13}	3	2.0088×10^{-11}	3	2.100

Table 6 Anisotropic creep properties of Ni-based DS superalloy

$A_{ m anisop} ({ m MPa}^{-np})$	nanisop	$q_{\rm aniso}$, (h ⁻¹)	$A_{\rm aniso} ({\rm MPa}^{-n}{\rm h}^{-1})$	n _{aniso}	$M_{\rm aniso} ({\rm MPa}^{-\chi}{\rm h}^{-1})$	Xaniso	$\phi_{ m aniso}$
2.8800×10^{-10}	3	1/8	$1.5929 imes 10^{-12}$	3	$1.5324 imes 10^{-11}$	3	2.889



Fig. 4 Creep deformation of DS Ni-based superalloy: (*a*) primary and secondary regimes and (*b*) all regimes

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Fig. 5 Damage evolution of DS Ni-based superalloy

The Young's modulus and Poisson ratio are not given; therefore, elastic damage will not be modeled. Using Sec. 3.2, the creep properties were determined and are given in Tables 5 and 6. Observation shows that $n_{pr} = n = \chi$ are equal for all orientations; therefore, the number of independent constants required (including elasticity) is reduced to 24.

Creep deformation and damage evolution simulations of these experiments were performed and shown in Figs. 4 and 5. The constitutive model accurately models both creep strain and damage evolution.

5 Conclusion

In conclusion, an isotropic and anisotropic multistage creep damage constitutive model has been developed. Taking cues from the principle of strain equivalence and the hypothesis of elastic energy equivalence, an elastic damage formulation has been developed. A method to determine the required material properties has been outlined in detail. Creep deformation data for copper and a DS Ni-based superalloy was obtained from literature and the relevant constitutive models applied. Results show that the constitutive models accurately model creep deformation and damage evolution for both materials. Future work will focus on developing an appropriate critical damage criterion for anisotropic materials such that failure is reached based on some critical damage equivalence. Alternatively, a creep rupture loci could be developed based on extensive mechanical testing.

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